

NON-LINEAR PLANK PROBLEMS AND POLYNOMIAL INEQUALITIES

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ABSTRACT. We study lower bounds for the norm of the product of polynomials and their applications to the so called *plank problem*. We are particularly interested in polynomials on finite dimensional Banach spaces, in which case our results improve previous works for large numbers of polynomials.

INTRODUCTION

The problem of finding lower bounds for the product of polynomials has been studied in several situations, considering a wide variety underlying spaces and norms. On a Banach space X , our study focuses on finding the best constant M such that, for any set of continuous scalar polynomials P_1, \dots, P_n over X , of some prescribed degrees, the following inequality holds

$$(1) \quad \|P_1\| \cdots \|P_n\| \leq M \|P_1 \cdots P_n\|.$$

In [6], C. Benítez, Y. Sarantopoulos and A. Tonge proved that, for continuous polynomials of degrees k_1, \dots, k_n , inequality (1) holds for every complex Banach space with constant

$$(2) \quad M = \frac{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}{k_1^{k_1} \cdots k_n^{k_n}}.$$

The authors also showed that this is the best universal constant, since there are polynomials on ℓ_1 for which we have equality. For complex Hilbert spaces and homogeneous polynomials, the second named author proved in [16] that the optimal constant is

$$(3) \quad M = \sqrt{\frac{(k_1 + \cdots + k_n)^{k_1 + \cdots + k_n}}{k_1^{k_1} \cdots k_n^{k_n}}},$$

when the dimension of the space is at least the number of polynomials. Using a complexification argument it is easy to find a constant for real Hilbert space from (3). But, in this context, a stronger result was given in [14] by D. Malicet, I. Nourdin, G. Peccati

and G. Poly: for real d –dimensional Hilbert spaces and homogeneous polynomials, (1) holds with constant

$$(4) \quad M = \sqrt{\frac{2^{k_1+\dots+k_n} \Gamma(k_1 + \dots + k_n + \frac{d}{2})}{\Gamma(\frac{d}{2}) k_1! \dots k_n!}}.$$

For homogeneous polynomials on L_p spaces or on the Schatten classes \mathcal{S}_p , with $1 \leq p \leq 2$, in [10] we showed that the optimal constant is

$$(5) \quad M = \sqrt[p]{\frac{(k_1 + \dots + k_n)^{k_1+\dots+k_n}}{k_1^{k_1} \dots k_n^{k_n}}}.$$

Some further references on this and related problems, where other polynomial norms and different Banach spaces are considered, are [1, 5, 8, 17, 18].

In this work we aim to find better constants for finite dimensional spaces. We are able to improve some of the previous results when the number of polynomials is much larger than the dimension of the space (see Theorem 1.1 and the comments following it). We also obtain specific bounds for Hilbert spaces.

As an application of the different versions of (1), we address a polynomial version of the *plank problem*. The following problem was posed by Alfred Tarski in the early 1930's [20, 21].

Given a convex body $K \subset \mathbb{R}^d$, of minimal width 1, when K is covered by n parallel strips or planks with widths a_1, \dots, a_n , is it true that $\sum_{i=1}^n a_i \geq 1$?

The solution to this problem was given by T. Bang [4], who also presented the following related question.

When a convex body is covered by planks, is it true that the sum of the relative widths is greater than or equal to 1?

This question remains unanswered in the general case, but for centrally symmetric convex bodies the solution was given by K. Ball in [2], where he proved (slightly more than) the following.

If $(\phi_j)_{j \in \mathbb{N}}$ is a sequence of norm 1 linear functionals on a (real) Banach space X and $(a_j)_{j \in \mathbb{N}}$ is a sequence of non-negative numbers whose sum

is less than 1, then there is a point \mathbf{z} in the unit ball of X for which $|\phi_j(\mathbf{z})| \geq a_j$ for every $j \in \mathbb{N}$.

To realize that this is a sharp result, it is enough to consider $X = \ell_1$ and the vectors of the standard basis of its dual, ℓ_∞ . However, when we restrict ourselves to some special Banach spaces and functionals, better constraints can be found. For example, given $\{\phi_1, \dots, \phi_n\}$ a set of orthonormal linear functionals defined on a Hilbert space \mathcal{H} , it is clear that for any set of real numbers $\{a_1, \dots, a_n\}$ such that $\sum_{j=1}^n a_j^2 \leq 1$, it is possible to find a vector \mathbf{z} in the unit ball of \mathcal{H} satisfying $|\phi_j(\mathbf{z})| \geq a_j$ for $j = 1, \dots, n$. This is not necessary true if we choose other sets of unit functionals on a real Hilbert space. For complex Hilbert spaces the situation is better, as K. Ball showed in [3]:

If $(\phi_j)_{j \in \mathbb{N}}$ is a sequence of norm 1 linear functionals on a complex Hilbert space \mathcal{H} and $(a_j)_{j \in \mathbb{N}}$ is a sequence of non-negative numbers satisfying $\sum_{j=1}^\infty a_j^2 = 1$, then there is a unit vector $\mathbf{z} \in \mathcal{H}$ for which $|\phi_j(\mathbf{z})| \geq a_j$ for every $j \in \mathbb{N}$.

This result implies the following inequality: let $S_{\mathcal{H}}$ denote the unit sphere of \mathcal{H} , then for any set of vectors $\{\omega_1, \dots, \omega_n\} \subset S_{\mathcal{H}}$ we have

$$\sup_{\mathbf{z} \in S_{\mathcal{H}}} |\langle \mathbf{z}, \omega_1 \rangle \cdots \langle \mathbf{z}, \omega_n \rangle| \geq \frac{1}{\sqrt{n^n}}.$$

The last inequality was proved by J. Arias-de-Reyna [1] a few years before Ball's article using a different technique. It is related to the lower bounds for the norm of the product of polynomials mentioned above.

Using results from [6, 16], A. Kavadiklis and S. G. Kim [12] studied a plank type problem for polynomials on Banach spaces. In this article we exploit the lower bounds for the product of polynomials given in [6, 10, 16], as well as the lower bounds we study in Section 1, to address this kind of problems.

By a polynomial plank problem we mean to give conditions such that, for any set of positive real numbers a_1, \dots, a_n , fulfilling them, and any set of continuous scalar polynomials P_1, \dots, P_n over a Banach space X , of degrees k_1, \dots, k_n , there is a vector \mathbf{z} in the unit ball of X for which $|P_j(\mathbf{z})| \geq a_j^{k_j}$ for $j = 1, \dots, n$.

1. MAIN RESULTS

We begin this section with some notation. Given a Banach space X , B_X denotes the unit ball of X . Recall that a function $P : X \rightarrow \mathbb{K}$ is a continuous k -homogeneous polynomial if there is a continuous k -linear function $T : X \times \cdots \times X \rightarrow \mathbb{K}$ for which $P(\mathbf{z}) = T(\mathbf{z}, \dots, \mathbf{z})$. A function $Q : X \rightarrow \mathbb{K}$ is a continuous polynomial of degree k if $Q = \sum_{l=0}^k Q_l$, with Q_0 a constant, Q_l an l -homogeneous polynomial ($1 \leq l \leq k$) and $Q_k \neq 0$. The norm of a polynomial Q is defined as

$$\|Q\| = \sup_{\mathbf{z} \in B_X} |Q(\mathbf{z})|.$$

1.1. Lower bounds for the product of polynomials. Our main result concerning lower bounds for the product of polynomials on finite dimensional spaces is the following.

Theorem 1.1. *Let X be a d -dimensional Banach space over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and P_1, \dots, P_n scalar polynomials of degrees k_1, \dots, k_n over X . Then*

$$(6) \quad \|P_1\| \cdots \|P_n\| \leq \frac{(C_{\mathbb{K}} 4ed)^{\sum_{i=1}^n k_i}}{2^{\frac{n}{C_{\mathbb{K}}}}} \|P_1 \cdots P_n\|,$$

where $C_{\mathbb{R}} = 1$ and $C_{\mathbb{C}} = 2$.

Let us compare this result with the results on this problem mentioned in the Introduction. The inequality (6) is better than (2) for finite dimensional spaces provided that the number of polynomials is large enough. The same is true when we apply (6) to homogeneous polynomials on finite dimensional L_p spaces, with $1 \leq p \leq 2$, and compare it with (3) and (5).

Comparing (6) and (4) for finite dimensional real Hilbert spaces it is not as straightforward. In this case, which bound is better depends on the particular setting. For example, when we consider n homogeneous polynomials of the same degree k on a d dimensional real Hilbert space, for large values of k we see that (6) is better than (4) if $n > 8e^2 d^2$, and that (4) is better than (6) if $n < 8e^2 d^2$. On the other hand, if we fix k , (6) is better than (4) for n large enough.

Theorem 1.1 is, in some sense we now explain, sharp as a general result. A look at Theorem 1.1 suggests an extension of the problem of finding the linear polarization

constant of a space X (see [15, 17]). We first define $\mathbf{M}_n(X)$ as the optimal constants such that, for any set of continuous scalar polynomials P_1, \dots, P_n over X , of degrees k_1, \dots, k_n , the following inequality holds

$$\|P_1\| \cdots \|P_n\| \leq \mathbf{M}_n(X)^{\sum_{i=1}^n k_i} \|P_1 \cdots P_n\|.$$

Then we set

$$(7) \quad \mathbf{M}(X) = \overline{\lim}_{n \rightarrow \infty} \mathbf{M}_n(X).$$

As a consequence of Theorem 1.1 we have

$$\mathbf{M}(X) \leq C_{\mathbb{K}} 4ed.$$

Let us see that taking $X = \ell_1^d$ the linear growth in d can be attained, which shows the sharpness of Theorem 1.1. In this case, for $n > d$, we define for $i = 1, \dots, n$ the homogeneous polynomial P_i on $\ell_1^d(\mathbb{K})$ by

$$P_i(z_1, \dots, z_d) = \begin{cases} z_i^{k(n-d+1)} & \text{if } i < d \\ z_d^k & \text{if } i \geq d \end{cases}$$

This is a set of n polynomials and, using Lagrange multipliers (see Lemma 2.3 from [10]), it is easy to see the following:

$$\begin{aligned} \|P_1 \cdots P_n\| &= \frac{1}{d^{k(n-d+1)}} \|P_1\| \cdots \|P_{d-1}\| \|P_d \cdots P_n\| \\ &= \frac{1}{d^{\sum_{i=1}^n \deg(P_i)}} \|P_1\| \cdots \|P_{d-1}\| \|P_d\| \cdots \|P_n\|. \end{aligned}$$

Therefore, $\mathbf{M}_n(\ell_1^d(\mathbb{K})) \geq d$, and thus $\mathbf{M}(\ell_1^d(\mathbb{K})) \geq d$. Then, we conclude that $\mathbf{M}(\ell_1^d(\mathbb{K}))$ increases at the same rate as d . Note that k was arbitrary, so we cannot improve this growth rate by fixing the degrees of the polynomials.

For finite dimensional Hilbert spaces, we have the following result, which gives better bounds than those of Theorem 1.1.

Proposition 1.2. *Let \mathcal{H} be a d dimensional (real or complex) Hilbert space and P_1, \dots, P_n scalar homogeneous polynomials of degrees k_1, \dots, k_n over \mathcal{H} . Then*

$$(8) \quad \|P_1\| \cdots \|P_n\| \leq \left(\frac{e^{H_d C_{\mathbb{K}}}}{4} \right)^{\sum_{i=1}^n k_i} \|P_1 \cdots P_n\|,$$

where $C_{\mathbb{R}} = 1$, $C_{\mathbb{C}} = 2$ and H_d stands for the d th harmonic number $\sum_{k=1}^d \frac{1}{k}$.

To see that this results improves Theorem 1.1 applied to Hilbert spaces it is enough to see that

$$\frac{e^{H_d C_{\mathbb{K}}/4}}{C_{\mathbb{K}} 2ed} < 1.$$

This follows from the fact that the sequence $H_l - \log(l)$ decreases, hence

$$\frac{e^{H_d C_{\mathbb{K}}/4}}{C_{\mathbb{K}} 2ed} \leq \frac{e^{H_1/4}}{2e} = \frac{1}{8}.$$

Finally, using that every homogeneous polynomial $P : \mathbb{C}^2 \rightarrow \mathbb{C}$ is the product of linear functions (see [19], Lemma 3.3.6), we obtain the following estimate for 2-dimensional complex Banach spaces.

Proposition 1.3. *Let X be a complex 2-dimensional Banach space and P_1, \dots, P_n scalar homogeneous polynomials of degrees k_1, \dots, k_n over X . Then*

$$\|P_1\| \cdots \|P_n\| \leq \mathbf{c}_{\mathbf{k}}(X) \|P_1 \cdots P_n\|,$$

where $\mathbf{k} = \sum_{j=1}^n k_j$ and $\mathbf{c}_{\mathbf{k}}(X)$ is the \mathbf{k} -th linear polarization constant of X .

This last result can be used to relate the linear polarization constant $\mathbf{c}(X)$ with the constant $\mathbf{M}(X)$ defined in (7).

Corollary 1.4. *Let X be a complex 2-dimensional Banach space, then*

$$\mathbf{M}(X) = \mathbf{c}(X).$$

Proof. The inequality

$$\mathbf{M}(X) \geq \mathbf{c}(X)$$

is immediate —and holds for any Banach space— since $\mathbf{M}_n(X) \geq \mathbf{c}_n(X)^{\frac{1}{n}}$.

For the other inequality, by Proposition 1.3, we have

$$\begin{aligned} \mathbf{M}_n(X) &\leq \sup\{\mathbf{c}_{\mathbf{k}}(X)^{\frac{1}{\mathbf{k}}} : k_1, \dots, k_n \in \mathbb{N}, \mathbf{k} = \sum_{j=1}^n k_j\} \\ &= \sup\{\mathbf{c}_l(X)^{\frac{1}{l}} : l \geq n\}, \end{aligned}$$

therefore

$$\mathbf{M}(X) \leq \lim_{l \rightarrow \infty} \sup\{\mathbf{c}_l(X)^{\frac{1}{l}} : l \geq n\} = \mathbf{c}(X). \quad \square$$

1.2. The polynomial plank problem. Our first main plank type result, and our most general one, is the following.

Theorem 1.5. *Let X be a complex Banach space and $P_1, \dots, P_n : X \rightarrow \mathbb{C}$ be norm one polynomials of degrees k_1, \dots, k_n . Given $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ satisfying $\sum_{i=1}^n a_i < \frac{1}{n^{n-1}}$, there is $\mathbf{z}_0 \in B_X$ such that*

$$|P_i(\mathbf{z}_0)| \geq a_i^{k_i} \text{ for } i = 1, \dots, n.$$

Moreover, if X is finite dimensional, this also holds for $\sum_{i=1}^n a_i = \frac{1}{n^{n-1}}$.

The proof of this theorem will make use of the constant provided in (2). Although the constant (2) is optimal for general complex Banach spaces, in some cases better constants have been obtained. As we have already mentioned, if X is the complex Banach space $L_p(\mu)$ or \mathcal{S}_p , with $1 \leq p \leq 2$, we have (5). Then, using (5) instead of (2), we obtain the following plank result.

Theorem 1.6. *Let X be the complex Banach space $L_p(\mu)$ or \mathcal{S}_p , with $1 \leq p \leq 2$, and $P_1, \dots, P_n : X \rightarrow \mathbb{C}$ be norm one homogeneous polynomials of degrees k_1, \dots, k_n . Given $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ satisfying $\sum_{i=1}^n a_i^p < \frac{1}{n^{n-1}}$, there is $\mathbf{z}_0 \in B_X$ such that*

$$|P_i(\mathbf{z}_0)| \geq a_i^{k_i} \text{ for } i = 1, \dots, n.$$

Moreover, if X is finite dimensional, this also holds for $\sum_{i=1}^n a_i^p = \frac{1}{n^{n-1}}$.

As pointed out before, the constants provided in (2) and (5) are optimal for infinite dimensional spaces or, in the case of the L_p spaces, when the dimension of the underlying spaces are at least n (the number of polynomials). Then, a natural next step is to use the sharper inequalities obtained for finite dimensional spaces in Theorem 1.1. To do this, we need the following.

Proposition 1.7. *Let X be a finite dimensional Banach space, n a natural number and suppose we have a positive constant $K < \frac{1}{\sqrt[n]{ne^2}}$ such that for any set $P_1, \dots, P_n : X \rightarrow \mathbb{C}$ of norm one polynomials we have*

$$\|P_1 \cdots P_n\| \geq K^{\sum_{i=1}^n k_i} \|P_1\| \cdots \|P_n\|,$$

where k_1, \dots, k_n are the degrees of the polynomials. Then, given $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$, with $\sum_{i=1}^n a_i \leq nK^n$, there is $\mathbf{z}_0 \in B_X$ such that

$$|P_i(\mathbf{z}_0)| \geq a_i^{k_i} \text{ for } i = 1, \dots, n.$$

Then, combining Proposition 1.7 and Theorem 1.1, we obtain the following plank type result for polynomials on finite dimensional spaces.

Proposition 1.8. *Let X be a d -dimensional Banach space over \mathbb{K} , $C_{\mathbb{K}}$ as in Theorem 1.1 and $P_1, \dots, P_n : X \rightarrow \mathbb{K}$ a set of norm one polynomials of degrees k_1, \dots, k_n . Given $a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ satisfying $\sum_{i=1}^n a_i \leq n \left(\frac{1}{C_{\mathbb{K}}^{4ed}} \right)^n$, there is $\mathbf{z}_0 \in B_X$ such that*

$$|P_i(\mathbf{z}_0)| \geq a_i^{k_i} \text{ for } i = 1, \dots, n.$$

Remark 1.9. It is natural to compare the plank type results described in this article to previous work. First, it is easy to see that for linear functions (i.e. homogeneous polynomials of degree one), we are far from recovering the optimal results of K. Ball on the plank problem. On the other hand, the value of our results relies on the generality in which they can be stated. They can be applied for polynomials of arbitrary (and different) degrees, and a large range of positive numbers a_1, \dots, a_n . Moreover, most of them also work for non homogeneous polynomials. In this way, we extend, and sometimes improve, previous work in the subject. For example, Theorem 5 of [12] can be recovered from Theorem 1.5 and Theorem 1.6 as a particular case, taking polynomials of the same degrees, all the scalars with the same value, etc.

2. THE PROOFS OF THE LOWER BOUNDS FOR PRODUCTS OF POLYNOMIALS

2.1. Proof of Theorem 1.1. In order to prove Theorem 1.1 we will use Remez type inequalities for polynomials in several variables. The objective of Remez type inequalities is to give bounds for classes of functions over some fixed set, given that the modulus of the functions is bounded on some subset of prescribed measure. For example, the original inequality of Remez states the following.

Take $a > 0$ and a polynomial $P : [-1, 1 + a] \rightarrow \mathbb{R}$ of degree k such that

$$\sup_{t \in V} |P(t)| \leq 1$$

for some measurable subset $V \subseteq [-1, 1+a]$, with $|V| \geq 2$. Then

$$\sup_{t \in [-1, 1+a]} |P(t)| \leq \sup_{t \in [-1, 1+a]} |T_k(t)|,$$

where T_k stands for the Chebyshev polynomial of degree k .

This inequality, combined with some properties of the Chebyshev polynomials, produces the following corollary, which most applications of Remez inequality use.

Corollary 2.1. *Let $P : \mathbb{R} \rightarrow \mathbb{R}$ be a polynomial of degree k , $I \subset \mathbb{R}$ be an interval and $V \subseteq I$ an arbitrary measurable set of strictly positive measure, then*

$$(9) \quad \sup_{t \in I} |P(t)| \leq \left(\frac{4|I|}{|V|} \right)^k \sup_{t \in V} |P(t)|.$$

We are interested in inequalities similar to (9), but for polynomials on several variables. Y. Brudnyi and M. Ganzburg studied Remez type inequalities for polynomials on several variables in [9]. As the original result of Remez, they stated their main result in terms of the Chebyshev polynomials.

Theorem 2.2. *Let X be a d -dimensional real space, λ a positive number and $P : X \rightarrow \mathbb{R}$ a polynomial of degree k such that*

$$\sup_{t \in V} |P(t)| \leq 1$$

for some measurable subset $V \subseteq B_X$, with $\mu(V) \geq \lambda$, where μ is the normalized Lebesgue measure over B_X . Then

$$\|P\| \leq T_k \left(\frac{1 + (1 - \lambda)^{\frac{1}{d}}}{1 - (1 - \lambda)^{\frac{1}{d}}} \right).$$

Just as in the applications of the original Remez inequality, rather than using Theorem 2.2, we will use the next proposition (see inequality (8) from [9]), which is a corollary from the main result of [9].

Proposition 2.3. *Let X be a d -dimensional real space and $P : X \rightarrow \mathbb{R}$ a polynomial of degree k . Given any Lebesgue measurable subset $V \subseteq B_X$ with $\mu(V) > 0$, we have*

$$\sup_{z \in B_X} |P(z)| \leq \frac{1}{2} \left(\frac{4d}{\mu(V)} \right)^k \sup_{z \in V} |P(z)|,$$

where μ is the normalized Lebesgue measure over B_X .

As an immediate consequence of this result, we have the following inequality (see inequality (14) from [9]). If P is a norm one polynomial of degree k over a finite d -dimensional Banach space X , then

$$(10) \quad \mu(\{\mathbf{z} \in B_X : |P(\mathbf{z})| \leq t\}) \leq 4d \left(\frac{t}{2}\right)^{\frac{1}{k}},$$

for any $0 < t < 1$.

Finally, we will need to use the following lemma.

Lemma 2.4. *Let $P : X \rightarrow \mathbb{R}$ be a norm one polynomial of degree k . Then*

$$\int_0^{+\infty} \mu(\{\mathbf{z} \in B_X : |P(\mathbf{z})| \leq e^{-t}\}) dt \leq -\ln\left(\frac{2}{(4d)^k}\right) + k,$$

where μ is the normalized Lebesgue measure over B_X .

Proof. To simplify notation let us write

$$V_t = \{\mathbf{z} \in B_X : |P(\mathbf{z})| \leq e^{-t}\}.$$

Then, using the Remez type inequality (10), we have

$$\begin{aligned} \int_0^{+\infty} \mu(V_t) dt &= \int_0^{-\ln\left(\frac{2}{(4d)^k}\right)} \mu(V_t) dt + \int_{-\ln\left(\frac{2}{(4d)^k}\right)}^{+\infty} \mu(V_t) dt \\ &\leq \int_0^{-\ln\left(\frac{2}{(4d)^k}\right)} 1 dt + \int_{-\ln\left(\frac{2}{(4d)^k}\right)}^{+\infty} 4d \frac{e^{-\frac{t}{k}}}{2^{\frac{1}{k}}} dt \\ &= -\ln\left(\frac{2}{(4d)^k}\right) + \frac{4d}{2^{\frac{1}{k}}} (-k) e^{-\frac{t}{k}} \Big|_{-\ln\left(\frac{2}{(4d)^k}\right)}^{+\infty} \\ &= -\ln\left(\frac{2}{(4d)^k}\right) + k. \quad \square \end{aligned}$$

Proof of Theorem 1.1. Given $P_1, \dots, P_n : X \rightarrow \mathbb{K}$ polynomials of degree k_1, \dots, k_n , we have to prove that

$$\|P_1 \cdots P_n\| \geq \frac{2^{\frac{n}{C_{\mathbb{K}}}}}{(C_{\mathbb{K}} 4ed)^{\sum_{i=1}^n k_i}} \|P_1\| \cdots \|P_n\|.$$

We start with the real case. We may assume all the polynomials have norm one. Using Lemma 2.4 we have:

$$\begin{aligned}
\ln(\|P_1 \cdots P_n\|) &= \sup_{\mathbf{z} \in B_X} \ln \left(\prod_{i=1}^n |P_i(\mathbf{z})| \right) \\
&\geq \int_{B_X} \ln \left(\prod_{i=1}^n |P_i(\mathbf{z})| \right) d\mu(\mathbf{z}) \\
&= \sum_{i=1}^n \int_{B_X} \ln |P_i(\mathbf{z})| d\mu(\mathbf{z}) \\
&= - \sum_{i=1}^n \int_{B_X} -\ln |P_i(\mathbf{z})| d\mu(\mathbf{z}) \\
&= - \sum_{i=1}^n \int_0^{+\infty} \mu(\{\mathbf{z} \in B_X : |P_i(\mathbf{z})| \leq e^{-t}\}) dt \\
&\geq \sum_{i=1}^n \ln \left(\frac{2}{(4d)^{k_i}} \right) - k_i.
\end{aligned}$$

We have used the fact that

$$\begin{aligned}
\int_{B_X} -\ln |P_i(\mathbf{z})| d\mu(\mathbf{z}) &= \int_0^{+\infty} \mu(\{\mathbf{z} \in B_X : -\ln |P_i(\mathbf{z})| \geq t\}) dt \\
&= \int_0^{+\infty} \mu(\{\mathbf{z} \in B_X : |P_i(\mathbf{z})| \leq e^{-t}\}) dt
\end{aligned}$$

which is an easy application of Funini's theorem.

Therefore

$$\begin{aligned}
\|P_1 \cdots P_n\| &\geq \exp \left\{ \sum_{i=1}^n \ln \left(\frac{2}{(4d)^{k_i}} \right) - k_i \right\} \\
&= \prod_{i=1}^n \frac{2}{(4d)^{k_i}} \frac{1}{e^{k_i}} \\
&= \frac{2^n}{(4de)^{\sum_{i=1}^n k_i}},
\end{aligned}$$

as desired.

To prove the complex case we will use the real case. Let X be a d -dimensional complex Banach space and $P_1, \dots, P_n : X \rightarrow \mathbb{C}$ polynomials of degree k_1, \dots, k_n . Take Y the $2d$ -dimensional real Banach space obtained from thinking X as a real space, and consider the polynomials $Q_1, \dots, Q_n : Y \rightarrow \mathbb{R}$, of degrees $2k_1, \dots, 2k_n$, defined as

$$Q_i(\mathbf{z}) = |P_i(\mathbf{z})|^2, \quad i = 1, \dots, n.$$

Applying inequality (6) for polynomials on a real Banach space to these polynomials we obtain

$$\begin{aligned} \|P_1\|^2 \cdots \|P_n\|^2 \frac{2^n}{(8de)^{\sum_{i=1}^n 2k_i}} &= \|Q_1\| \cdots \|Q_n\| \frac{2^n}{(8de)^{\sum_{i=1}^n 2k_i}} \\ &\leq \|Q_1 \cdots Q_n\| = \|P_1 \cdots P_n\|^2, \end{aligned}$$

which ends the proof. \square

2.2. Proof of Proposition 1.2. A cornerstone on the proof of Theorem 1.1 was the use of the Remez type inequality (10) to obtain Lemma 2.4. But when we restrict ourselves to homogeneous polynomials over Hilbert spaces we can prove the following sharper lemma (see [19], Lemma 3.3.4).

Lemma 2.5. *Let $P : \ell_2^d(\mathbb{R}) \rightarrow \mathbb{R}$ be a norm one homogeneous polynomial of degree k . Then*

$$\int_0^{+\infty} \mu(\{z \in B : |P(z)| \leq e^{-t}\}) \leq k(\ln(4) + H_d),$$

where H_d stands for the d th harmonic number.

Then, if we replace Lemma 2.4 with Lemma 2.5 in the proof of Theorem 1.1, we obtain the proof of Proposition 1.2.

2.3. Proof of Proposition 1.3. We may assume P_1, \dots, P_n are norm one polynomials. By Lemma 3.3.6 from [19], we know that

$$P_i = L_i \varphi_{i,1} \cdots \varphi_{i,k_i} \text{ for } i = 1, \dots, n,$$

where $\varphi_{i,j}$ are norm one linear functions and

$$L_i = \frac{1}{\|\varphi_{i,1} \cdots \varphi_{i,k_i}\|} \geq 1.$$

Then, by definition of the \mathbf{k} -th linear polarization constant, we have

$$\begin{aligned}
\|P_1 \cdots P_n\| &= \left\| \prod_{i=1}^n \left(L_i \prod_{j=1}^{k_i} \varphi_{i,j} \right) \right\| \\
&= \left(\prod_{i=1}^n L_i \right) \left\| \prod_{i=1}^n \prod_{j=1}^{k_i} \varphi_{i,j} \right\| \\
&\geq \left\| \prod_{i=1}^n \prod_{j=1}^{k_i} \varphi_{i,j} \right\| \\
&\geq \frac{1}{\mathbf{c}_k(X)}
\end{aligned}$$

which ends the proof.

3. PROOF OF THE PLANK TYPE RESULTS

3.1. Proof of Theorems 1.5 and 1.6. For the proof of Theorems 1.5 and 1.6 we need the following two technical lemmas.

Lemma 3.1. *Given n positive integers k_1, \dots, k_n , the set*

$$(11) \quad \left\{ \frac{1}{\sum_{i=1}^n k_i r_i} (k_1 r_1, \dots, k_n r_n) : r_1, \dots, r_n \in \mathbb{N} \right\}$$

is dense in $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$.

Proof. Take a set of rational numbers $t_1, \dots, t_n \in \mathbb{Q}$, with $t_i > 0$ ($i = 1, \dots, n$) and $\sum_{i=1}^n t_i = 1$. Write $t_i = \frac{q_i}{p}$ with $q_1, \dots, q_n \in \mathbb{N}$ and $p = \sum_{i=1}^n q_i$. Let $M = \prod_{i=1}^n k_i$ and take r_i such that $k_i r_i = q_i M$. Then

$$\frac{1}{\sum_{i=1}^n k_i r_i} (k_1 r_1, \dots, k_n r_n) = \frac{1}{\sum_{i=1}^n q_i M} (q_1 M, \dots, q_n M) = \left(\frac{q_1}{p}, \dots, \frac{q_n}{p} \right).$$

Now, the density of rational numbers gives the desired result. \square

Lemma 3.2. *Given $b_1, \dots, b_n \in \mathbb{R}_{\geq 0}$, with $\sum_{i=1}^n b_i = \frac{1}{n^{n-1}}$, there is an element $(t_1, \dots, t_n) \in \mathbb{R}_{>0}^n$ such that*

$$\sum_{i=1}^n t_i = 1 \quad \text{and} \quad t_1^{t_1} \cdots t_n^{t_n} \geq b_i^{t_i} \text{ for } i = 1, \dots, n.$$

Proof. For $n = 1$ we can take $t_1 = 1$ and for $n = 2$ we can take $t_1 = t_2 = \frac{1}{2}$. Let us assume $n \geq 3$. For each i let us define $\tilde{t}_i = -\frac{c}{\ln(b_i)}$, with $c > 0$ such that

$$(12) \quad \prod_{j=1}^n \tilde{t}_j = e^{-c}.$$

If we show $\sum_{j=1}^n \tilde{t}_j \leq 1$, and define $t_i = \frac{1}{\sum_{j=1}^n \tilde{t}_j} \tilde{t}_i$, then we have

$$\prod_{j=1}^n t_j^{t_j} = \frac{1}{\sum_{j=1}^n \tilde{t}_j} \left(\prod_{j=1}^n \tilde{t}_j^{\tilde{t}_j} \right)^{\frac{1}{\sum_{j=1}^n \tilde{t}_j}} \geq \prod_{j=1}^n \tilde{t}_j^{\tilde{t}_j} = e^{-c} = b_i^{\tilde{t}_i} \geq b_i^{t_i}.$$

Let us see this. Define $a_i = \frac{-1}{\ln(b_i)}$. Since each $b_i \leq \frac{1}{n^{n-1}} < \frac{1}{e^2}$ and the map $x \mapsto \frac{-1}{\ln(x)}$ is concave on $(0, \frac{1}{e^2})$, by Jensen's inequality, we have

$$(13) \quad \sum_{i=1}^n a_i \leq \frac{1}{\ln(n)}.$$

From (12) we have

$$\ln(c) = \frac{-1 - \sum_{i=1}^n a_i \ln(a_i)}{\sum_{i=1}^n a_i}.$$

Now, using that $x \mapsto x \ln(x)$ is convex on $\mathbb{R}_{>0}$, Jensen's inequality and (13) we obtain

$$\begin{aligned} \frac{-1 - \sum_{i=1}^n a_i \ln(a_i)}{\sum_{i=1}^n a_i} &\leq \frac{-1 - \sum_{i=1}^n a_i (\ln(\sum_{i=1}^n a_i) - \ln(n))}{\sum_{i=1}^n a_i} \\ &= -\frac{1}{\sum_{i=1}^n a_i} - \ln\left(\sum_{i=1}^n a_i\right) + \ln(n) \\ (14) \quad &\leq -\frac{1}{1/\ln(n)} - \ln(1/\ln(n)) + \ln(n) \\ &= \ln(\ln(n)), \end{aligned}$$

where in (14) we use that the map $x \mapsto -\frac{1}{x} - \ln(x)$ is increasing on $(0, 1)$. Then $c \leq \ln(n)$ and therefore

$$\sum_{j=1}^n \tilde{t}_j = \sum_{j=1}^n c a_j \leq \ln(n) \sum_{j=1}^n a_j \leq 1$$

which ends the proof. \square

Note that the inequalities in Theorem 1.5 and (2) are exactly the ones in Theorem 1.6 and (5) when we take $p = 1$. With this in mind, it is easy to see that, if we put $p = 1$ in the proof of Theorem 1.6 below, we obtain the proof of Theorem 1.5.

Proof of Theorem 1.6. Choose $b_i > a_i^p$, $i = 1, \dots, n$, such that $\sum_{i=1}^n b_i = \frac{1}{n^{n-1}}$. By Lemma 3.2, we can take an element $(t_1, \dots, t_n) \in \mathbb{R}_{>0}^n$ with $\sum_{i=1}^n t_i = 1$ and

$$t_1^{t_1} \cdots t_n^{t_n} \geq b_i^{t_i} \text{ for } i = 1, \dots, n.$$

We claim that there is $\delta > 0$ such that for any positive integer N , we can choose $r = (r_1, \dots, r_n) \in \mathbb{N}^n$ so that, if we call $s_i = \frac{k_i r_i}{\sum_{j=1}^n k_j r_j}$, we have

$$s_i \geq \delta \quad \text{and} \quad s_1^{s_1} \cdots s_n^{s_n} \geq b_i^{s_i} \left(1 - \frac{1}{N}\right) \quad \text{for } i = 1, \dots, n.$$

In the notation we omit the dependence on N for simplicity. Indeed, using the convention that the function $t \mapsto t^t$ is one at $t = 0$, we have the finite family of continuous functions $x \mapsto x_1^{x_1} \cdots x_n^{x_n}$ and $x \mapsto b_i^{x_i}$, $i = 1, \dots, n$, defined on the compact set $\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$. Then, the density of the set (11) in this compact set is just what we need for finding r such that $s_1^{s_1} \cdots s_n^{s_n} \geq b_i^{s_i} (1 - \frac{1}{N})$. Since we have to take s_i close enough to t_i we also may assume $s_i \geq \frac{t_i}{2}$. Then, we define $\delta := \min \{\frac{t_i}{2} : i = 1, \dots, n\}$.

On the other hand, by (5), we have

$$\left\| \prod_{i=1}^n P_i^{r_i} \right\| \geq \left(\frac{\prod_{i=1}^n (k_i r_i)^{k_i r_i}}{(\sum_{i=1}^n k_i r_i)^{\sum_{i=1}^n k_i r_i}} \right)^{\frac{1}{p}}.$$

So, for all $N \in \mathbb{N}$, we can take $\mathbf{z}_N \in B_X$ such that

$$|(P_1(\mathbf{z}_N))^{r_1} \cdots (P_n(\mathbf{z}_N))^{r_n}| \geq \left(\frac{\prod_{i=1}^n (k_i r_i)^{k_i r_i}}{(\sum_{i=1}^n k_i r_i)^{\sum_{i=1}^n k_i r_i}} \right)^{\frac{1}{p}} \left(1 - \frac{1}{N}\right).$$

Since each polynomial has norm one, this gives for each $i = 1, \dots, n$:

$$|(P_i(\mathbf{z}_N))^{r_i}| \geq \left(\frac{\prod_{j=1}^n (k_j r_j)^{k_j r_j}}{(\sum_{j=1}^n k_j r_j)^{\sum_{j=1}^n k_j r_j}} \right)^{\frac{1}{p}} \left(1 - \frac{1}{N}\right).$$

Therefore,

$$\begin{aligned}
 |P_i(\mathbf{z}_N)| &\geq \left(\frac{\prod_{j=1}^n (k_j r_j)^{k_j r_j}}{(\sum_{j=1}^n k_j r_j)^{\sum_{j=1}^n k_j r_j}} \right)^{\frac{k_i}{k_i r_i p}} \left(1 - \frac{1}{N} \right)^{\frac{1}{r_i}} \\
 &= (s_1^{s_1} \cdots s_n^{s_n})^{\frac{k_i}{s_i p}} \left(1 - \frac{1}{N} \right)^{\frac{1}{r_i}} \\
 &\geq \left(b_i^{s_i} \left(1 - \frac{1}{N} \right) \right)^{\frac{k_i}{s_i p}} \left(1 - \frac{1}{N} \right) \\
 &= b_i^{\frac{k_i}{p}} \left(1 - \frac{1}{N} \right)^{\frac{k_i}{s_i p} + 1} \\
 (15) \quad &\geq b_i^{\frac{k_i}{p}} \left(1 - \frac{1}{N} \right)^{\frac{k_i}{\delta p} + 1}.
 \end{aligned}$$

But recall that $b_i > a_i^p$, $i = 1, \dots, n$. Since δ does not depend on N , we can take N large enough such that

$$b_i^{\frac{k_i}{p}} \left(1 - \frac{1}{N} \right)^{\frac{k_i}{\delta p} + 1} \geq a_i^{k_i},$$

which ends the proof of the general case.

For the finite dimensional case, we need to deal with the case $\sum_{i=1}^n a_i^p = \frac{1}{n^{n-1}}$. For this, we take $b_i = a_i^p$ and proceed as in the proof of the general case up to (15). We can take, by the finite dimension of our space, a limit point $\mathbf{z}_0 \in B_X$ of the sequence $\{\mathbf{z}_N\}_{N \in \mathbb{N}}$. Then, by continuity, we have

$$|P_i(\mathbf{z}_0)| \geq b_i^{\frac{k_i}{p}} = a_i^{k_i},$$

as desired. □

3.2. Proof of Proposition 1.7. The proof of Proposition 1.7 is analogous to the proof of Theorems 1.5 and 1.6, replacing Lemma 3.2 with the following lemma.

Lemma 3.3. *Let n be a natural number and $K \in \left(0, \frac{1}{\sqrt[n]{ne^2}}\right]$. Given non negative numbers b_1, \dots, b_n , with $\sum_{i=1}^n b_i = nK^n$, there is an element $(t_1, \dots, t_n) \in \mathbb{R}_{\geq 0}^n$ such that*

$$\sum_{i=1}^n t_i = 1 \quad \text{and} \quad K^{\frac{1}{t_i}} \geq b_i \quad \text{for } i = 1, \dots, n.$$

Proof. Let us first assume b_1, \dots, b_n are strictly positive and define $s_i := \frac{\ln(K)}{\ln(b_i)}$, or equivalently $b_i = K^{\frac{1}{s_i}}$. If we show that $\sum_{j=1}^n s_j \leq 1$, then we can take $t_i \geq s_i$ such that $\sum_{j=1}^n t_j = 1$ and, since $x \mapsto K^{\frac{1}{x}}$ is increasing, we have

$$K^{\frac{1}{t_i}} \geq K^{\frac{1}{s_i}} = b_i.$$

Let us see then, that $\sum_{j=1}^n s_j \leq 1$. The condition $\sum_{j=1}^n b_j = nK^n$ implies $b_i \leq \frac{1}{e^2}$ for each $i = 1, \dots, n$. Since the function $x \mapsto \frac{\ln(K)}{\ln(x)}$ is concave on $[0, \frac{1}{e^2}]$, using Jensen's inequality we have

$$\sum_{j=1}^n s_j = \sum_{j=1}^n \frac{\ln(K)}{\ln(b_j)} \leq n \frac{\ln(K)}{\ln\left(\frac{\sum_{j=1}^n b_j}{n}\right)} = 1.$$

If $b_{i_0} = 0$ for some i_0 , we define $s_i = 0$ whenever $b_i = 0$ and $s_i := \frac{\ln(K)}{\ln(b_i)}$ otherwise. Since in this case we do not have $b_1 = b_2 = \dots = b_n$, proceeding as in the previous case we obtain

$$\sum_{j=1}^n s_j < 1.$$

This allow us to take each t_i strictly greater than s_i (and, in particular, strictly positive as desired), satisfying $\sum_{j=1}^n t_j = 1$. We go on as above to obtain the result. \square

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